Journal of Statistical Physics, Vol. 58, Nos. 5/6, 1990

The *n*-Component Cubic Model and Flows: Subgraph Break-Collapse Method

A. C. N. de Magalhães[1] and J. W. Essam[2]

Received August 21, 1989

We specialize to the n-component cubic model the subgraph break-collapse method which we recently developed for the $Z(\lambda)$ model. The cubic model has less symmetry than the Potts model, for which the method was originally developed, but nevertheless it is still possible to reduce considerably the computational complexity of the general $Z(\lambda)$ model. Our recursive algorithm is similar, for $n = 2$, to the break-collapse method for the $Z(4)$ model proposed by Mariz and co-workers. It allows the exact calculation for the partition function and correlation functions for n-component cubic clusters, with n as a variable, without the need to examine all of the spin configurations. An important application is therefore in real-space renormalization-group calculations.

KEY WORDS: Cubic model; subgraph break-collapse method; graph theory.

1. INTRODUCTION

The n-component cubic model was introduced by Kim *et al.*[1] in the description of phase transitions in cubic rare-earth compounds which have sixfold-degenerate ground states (and hence correspond to $n = 3$). Aharony[2] generalized this model in order to include quadrupolar interactions, besides the dipolar ones. This extended cubic model, which we will henceforth call for simplicity the cubic model, is a discrete version of the continuous n-component spin model. Since its introduction, the cubic model has been studied by several methods.[3-10] It contains many interesting limiting cases (e.g., self-avoiding walks, spin-1/2 Ising model, the Ashkin–Teller model, and the Potts model) and for $n = 1$ and $n = 2$ it becomes identical to the Ising and $Z(4)$ models, respectively. For a general

[1] Centro Brasileiro de Pesquisas Fisicas (CBPF/CNPq), 22290-RJ, Rio de Janeiro, Brazil.
[2] Department of Mathematics, Royal Holloway and Bedford New College, Egham Hill, Egham, Surrey, TW20 OEX, England.

0022-4715/90/0300-1059$06.00/0

value of n, the cubic model is a particular case of the $Z(2n)$ model in which many values of the pair interaction energy become degenerate, leading to only three which are distinct.

In a recent paper on the $Z(\lambda)$ model[11] [entitled "The $Z(\lambda)$ model and flows" and herein referred to as ZF] we developed a recursive algorithm for the calculation of the exact partition function and pair correlation functions of $Z(\lambda)$ clusters. These clusters were represented by graphs, the vertices and edges of which represented, respectively, the atoms and the pair interactions between them. This technique, the subgraph break-collapse method (SBCM), is an extension of the SBCM for the Potts model[12] which we presented in paper III of the series of papers with the general title "Potts model and flows" (herein referred to as PF3). The SBCM for the $Z(\lambda)$ model is based on a number of equations—the "graph reduction equations," the proofs of which were given in ZF through the use of mod-λ flows in graphs. One of these equations, the effective break-collapse equation, relates the partition function and correlation functions for a graph G to those for the "broken" graphs, "collapsed" graphs, and graphs with "frozen edges." These graphs are obtained from G by respectively deleting, contracting, and fixing the value of the flow in a chosen edge f. The other graph reduction equations allow the calculation of the above-mentioned functions for articulated graphs and graphs in series or in parallel. The SBCM provides an efficient way of computing the partition function and the correlation functions by applying recursively the graph reduction equations, thereby avoiding the time-consuming summation over states.

An alternative method for calculating the above functions is the break-collapse method (BCM) of Mariz and co-workers.[13-15] The latter method differs from the SBCM in three main aspects: (i) it only replaces a subgraph which is a combination of edges in series and/or in parallel by a single effective edge, whereas the SBCM uses a more general subgraph replacement; (ii) its break-collapse equation contains graphs with "precollapsed" edges instead of "frozen" ones; (iii) with the exception of graphs with two vertices, the recursion terminates when all the edges of G are precollapsed rather than when just $c(G)$ of them are frozen [here $c(G)$ is the number of independent cycles in G]. In ZF precollapsed edges were shown to correspond to edges on which the flow can take on several values (namely 0, β, $\lambda-\beta$). Although, therefore, for $\lambda>4$, the BCM generates less graphs in each iteration than the SCBM, it was argued in ZF that for any λ the BCM is still less efficient than the SBCM. The reasons for this are twofold: (a) it needs more iterations; (b) for $\lambda>4$ the determination of the weight to be associated with a terminal graph (i.e., graphs with all edges precollapsed) is an enumeration problem whose computing time grows exponentially with the number of cycles in the graph. For $\lambda = 4$ a simple formula for the weight of a terminal graph is available.

Here we specialize the above SBCM to the n-component cubic model, taking advantage of the high degree of symmetry of its Hamiltonian. In particular, the effective break-collapse equation contains a sum of terms corresponding to the chosen edge f being frozen with values 0, 2, 4,..., $2n-2$. These terms can be naturally grouped together, leading to a single term which corresponds to a graph for which the flow on f must be even. We call such an edge "even-frozen," and for $n=2$ it becomes identical with the precollapsed edge introduced by Mariz *et al.*[13] in the $Z(4)$ algorithm. Unlike the BCM for the $Z(\lambda)$ model, in our algorithm for the cubic model the weights of the terminal graphs with all edges even-frozen are given by simple formulas *for any value of n*. Besides not having the inconvenience of the BCM mentioned in (b) above, our method allows the calculation of correlation functions for cubic clusters *for all values of n simultaneously* through a single application of the SBCM. The BCM has been successfuly applied in real-space renormalization-group (RG) calculations of phase diagrams and critical exponents (see ref. 15 and further references therein) and the SBCM described here can similarly be applied to the cubic model. It has already been applied[16] in obtaining the RG recursion relations used in the study of criticality in the n-component cubic antiferromagnet on the square lattice.

In Section 2 we introduce the model and summarize previous results concerning the partition function[17,18] and correlation functions (ZF) for the $Z(\lambda)$ model. In Section 3 we establish the relationship between the cubic model and the $Z(2n)$ model. We also prove that the equivalent vector transmissivity (from which one can calculate the correlation functions) has only two different components. The graph reduction equations of the SBCM are given in Section 4. In Section 5 we describe the SBCM algorithm and illustrate it by the example of the Wheatstone bridge graph. Finally, our conclusions are presented in Section 6.

2. MODEL AND REVIEW OF KNOWN RESULTS

In this section we define the model and summarize existing results[11,17,18] for the $Z(\lambda)$ model which will be needed in the development of the subsequent sections.

2.1. The Cubic Model

We consider the n-component cubic model for a graph G with vertex set V and edge set E. With each vertex i of V we associate an n-component vector which can point in one of the $2n$ directions (positive and negative) of the Cartesian axes in an n-dimensional space, i.e.,

$$\mathbf{S}_i = (\pm 1, 0, \ldots, 0) \text{ or } (0, \pm 1, 0, \ldots, 0) \text{ or } \ldots (0, 0, 0, \ldots, \pm 1) \tag{2.1}$$

The cubic model can be described by the following dimensionless Hamiltonian[2]:

$$\beta H(G) = -\sum_{e \in E} [nK_e \mathbf{S}_i \cdot \mathbf{S}_j + nL_e(\mathbf{S}_i \cdot \mathbf{S}_j)^2] \tag{2.2}$$

where $\beta = 1/k_B T$, and K_e and L_e are the respective dimensionless coupling constants associated with the dipolar and quadrupolar interactions between spins $\mathbf{S}_i$ and $\mathbf{S}_j$ located at the vertices i and j of the edge e. The sum in Eq. (2.2) is over all interacting spin pairs on G.

The Hamiltonian (2.2) may be written also in terms of an n-state Potts variable α_i ($\alpha_i = 0, 1, \ldots, n-1$) and an Ising spin variable σ_i ($\sigma_i = \pm 1$) as[2]

$$\beta H(G) = -\sum_{e \in E} [nK_e \sigma_i \sigma_j \delta(\alpha_i, \alpha_j) + nL_e \delta(\alpha_i, \alpha_j)] \tag{2.3}$$

which is a particular case of the (N_α, N_β) model (corresponding to $N_\beta = 2$ and $K_{1,1} = K_{1,0}$) introduced by Domany and Riedel.[6]

2.2. Known Results for the $Z(\lambda)$ Model

In ZF, a $Z(\lambda)$ cluster is represented by a graph G with vertex set V, edge set E, number of vertices v, and number of edges ε. With each vertex i of V is associated a state variable n_i which can take on one of the λ integer values $0, 1, \ldots, \lambda - 1$. The dimensionless Hamiltonian is given by

$$\beta H(G) = \sum_{e \in E} h_e(n_i - n_j) \tag{2.4}$$

where $n_i - n_j$ is calculated mod-λ and the pair interaction energy is independent of the ordering of i and j, so that

$$h_e(\lambda - \alpha) = h_e(\alpha) \tag{2.5}$$

The components $t_e(\alpha)$ of the λ-dimensional vector transmissivity t_e of the edge e are defined[19] by

$$t_e(\alpha) = \frac{1}{z_e} \sum_{\beta=0}^{\lambda-1} e^{2\pi i \alpha \beta/\lambda} e^{-h_e(\beta)} \qquad (\alpha = 0, 1, \ldots, \lambda - 1) \tag{2.6a}$$

where

$$z_e = \sum_{\alpha=0}^{\lambda-1} e^{-h_e(\alpha)} \tag{2.6b}$$

Of these λ-components, only $\lambda = [\lambda/2]$ (where $[\cdot]$ stands for the integer part) are independent, since $t_e(0) = 1$ and $t_e(\lambda - \alpha) = t_e(\alpha)$.

The partition function $Z(G)$ can be expressed[17,18] in terms of $t_e(\alpha)$ as

$$Z(G) = \lambda^{v-\varepsilon} \left(\prod_{e \in E} z_e \right) D(G) \tag{2.7a}$$

Here $D(G)$ is the generating function for flows given by

$$D(G) = \sum_{\varphi \in F(G)} \prod_{e \in E} t_e(\varphi(e)) \tag{2.7b}$$

where $\varphi(e)$ is the value of the mod-λ flow φ on the edge e, and $F(G)$ is the set of all mod-λ flows on G. Given an arbitrary directing of the edges $e \in E$, one can define a mod-λ flow (see, for example, ref. 20) as a function defined on E which assigns to each edge e one of the integer values $0, 1, \ldots, \lambda - 1$ subject to a "conservation condition" at each vertex $i \in V$, i.e., the sum of the inward flows minus the sum of the outward flows at i is zero mod-λ.

Pair correlation functions can normally be written as the thermal average of some function $f(n_1 - n_2)$ which depends only on the difference, mod-λ, of state variables n_1 and n_2. Here 1 and 2 refer to arbitrarily chosen vertices (called roots of the graph), and the thermal average can be Fourier decomposed as (ZF)

$$\langle f(n_1 - n_2) \rangle_{\text{thermal}} = \frac{1}{\lambda} \sum_{\alpha=0}^{\lambda-1} f_{\lambda-\alpha} T_\alpha(1, 2; G) \tag{2.8}$$

where

$$T_\alpha(1, 2; G) \equiv \langle e^{-2\pi i(n_1 - n_2)/\lambda} \rangle_{\text{thermal}} = \frac{N_\alpha(1, 2; G)}{D(G)} \tag{2.9a}$$

with

$$N_\alpha(1, 2; G) = \sum_{\varphi \in F_\alpha(G)} \prod_{e \in E} t_e(\varphi(e)) \tag{2.9b}$$

In (2.9b), $F_\alpha(G)$ is the set of all *rooted mod-λ α-flows*, i.e., of mod-λ flows subject to a fixed external flow α entering at root 1 and leaving at root 2.

Here $N(1, 2; G) = \{N_\alpha(1, 2; G),\ \alpha = 0, 1,..., \lambda - 1\}$ is called the *flow-vector*, although strictly speaking each of its components is a generating function for internal flows having a fixed external flow α in at 1 and out at 2. Notice that $N_0(1, 2; G)$ is exactly $D(G)$ given by (2.7b).

The vector $T(1, 2; G) = \{T_\alpha(1, 2; G),\ \alpha = 0, 1,..., \lambda - 1\}$ is called the *equivalent vector transmissivity* between the roots 1 and 2 of G, since it is equal to the vector transmissivity t_{eff} of a single *effective edge* between 1 and 2 having an equivalent Hamiltonian $h_{\text{eq}}(n_1 - n_2)$ given by

$$\mathrm{Tr}'\left\{\exp\left[-\sum_{e \in E} h_e(n_i - n_j)\right]\right\} = C \exp[-h_{\text{eq}}(n_1 - n_2)] \tag{2.10}$$

where C is a constant and Tr$'$ denotes the sum over all possible values of n_i for all vertices i different from the roots 1 and 2. The replacement of a cluster of atoms by a single effective edge connecting just two atoms with an effective interaction plays a fundamental role in real-space renormalization-group calculations and also, as we will see later on, in the SBCM.

In the case of the Potts model we have that

$$h_e(n_i - n_j) = \begin{cases} h_e(0) & \text{for} \quad n_i = n_j \\ \lambda K_e + h_e(0) & \text{for} \quad n_i \neq n_j \end{cases} \tag{2.11}$$

and, therefore, $t_e(\alpha) = t_e$ for any $\alpha \neq 0$, where t_e is the thermal transmissivity [Eq. (2.2) of PF3] used in many real-space renormalization-group calculations (see, for example, ref. 15). For the Potts model, Eq. (2.9b) reduces to

$$\begin{aligned} N_1(1, 2; G) = N_2(1, 2; G) = \cdots = N_{\lambda-1}(1, 2; G) &\equiv N(1, 2; G) \\ = \sum_{G' \subseteq G} F_{12}(\lambda, G') \prod_{e \in E'} t_e \end{aligned} \tag{2.12a}$$

and

$$N_0(1, 2; G) = D(G) = \sum_{G' \subseteq G} F(\lambda, G') \prod_{e \in E'} t_e \tag{2.12b}$$

where $F_{12}(\lambda, G')$ and $F(\lambda, G')$ are, respectively, the two-rooted and unrooted flow polynomials[20] of the partial graph G' of G. They correspond to the respective numbers of proper [i.e., $\varphi(e) \neq 0$ for all e] rooted mod-λ α-flows and unrooted flows.

3. THE TWO-COMPONENT EQUIVALENT VECTOR TRANSMISSIVITY

3.1. Relationship between the Cubic Model and the $Z(2n)$ Model

For $n=1$ and 2 the n-component cubic model is equal to the Ising and $Z(4)$ models, respectively. For general n, the cubic model is the particular case of the $Z(2n)$ model in which the pair interaction energies $h_e(\alpha)$ ($\alpha=0, 1,..., 2n-1$) become highly degenerate, namely

$$h_e(1)=h_e(2)=\cdots=h_e(n-1)=h_e(n+1)=\cdots=h_e(2n-1) \quad (3.1)$$

and where the energy differences are related to the dimensionless coupling constants K_e and L_e through

$$h_e(n)-h_e(0)=2nK_e \quad (3.2a)$$

and

$$h_e(1)-h_e(0)=n(K_e+L_e) \quad (3.2b)$$

Combining Eqs. (3.1) and (3.2) with the definition (2.6) of $t_e(\alpha)$, we arrive at only two components of the vector transmissivity which are different:

$$t_e(\alpha)=\frac{1-e^{-2nK_e}}{1+2(n-1)e^{-n(K_e+L_e)}+e^{-2nK_e}}\equiv t_e(1) \qquad (\alpha=1, 3,..., 2n-1) \quad (3.3a)$$

and

$$t_e(\alpha)=\frac{1-2e^{-n(K_e+L_e)}+e^{-2nK_e}}{1+2(n-1)e^{-n(K_e+L_e)}+e^{-2nK_e}}\equiv t_e(2) \qquad (\alpha=2, 4,..., 2n-2) \quad (3.3b)$$

The variables $t_e(1)$ and $t_e(2)$ are precisely the respective variables $\tilde{x}_\beta=\tilde{z}$ and $\tilde{x}_\alpha$ which appear in the model of Domany and Riedel[6] specialized to the cubic Hamiltonian. The two-dimensional vector $(t_e(1), t_e(2))$ is the vector thermal transmissivity of Tsallis *et al.*[10] used in their renormalization-group calculation of the critical frontier of the ferromagnetic cubic model on the square lattice.

3.2. The Two-Component Vector Equivalent Transmissivity

In this section we prove that, similarly to Eqs. (3.3), only two of the $2n-1$ components of the flow vector of the $Z(2n)$ model with $\alpha \neq 0$ are distinct in the case of the cubic model:

$$N_1(1, 2; G) = N_3(1, 2; G) = \cdots = N_{2n-1}(1, 2; G) \tag{3.4a}$$

and

$$N_2(1, 2; G) = N_4(1, 2; G) = \cdots = N_{2n-2}(1, 2; G) \tag{3.4b}$$

where $N_\alpha(1, 2; G)$ is defined in (2.9b) with

$$t_e(\varphi(e)) = t_e(1) \qquad \text{for odd} \quad \varphi(e) \tag{3.5a}$$

$$t_e(\varphi(e)) = t_e(2) \qquad \text{for even} \quad \varphi(e) \tag{3.5b}$$

and

$$t_e(0) = 1 \tag{3.5c}$$

The equalities (3.4) are related to the fact that, for the Potts model, $N_\alpha(1, 2; G)$ for $\alpha \neq 0$ is independent of the external flow α [see Eq. (2.12a)].

We first recall that, as shown in the appendix of ZF, one can generate the rooted mod-λ α-flows starting from the unrooted mod-λ flows. For this, one must choose a spanning tree τ on G which then determines a unique path θ which connects the roots 1 and 2. One can then construct a rooted mod-λ α-flow by adding, to each of the $\lambda^{c(G)}$ unrooted mod-λ flows, a flow Φ_α equal to α on the path θ from 1 to 2 and zero on all other edges. For example, for the graph G of Fig. 1a and the spanning tree τ of Fig. 1b, one

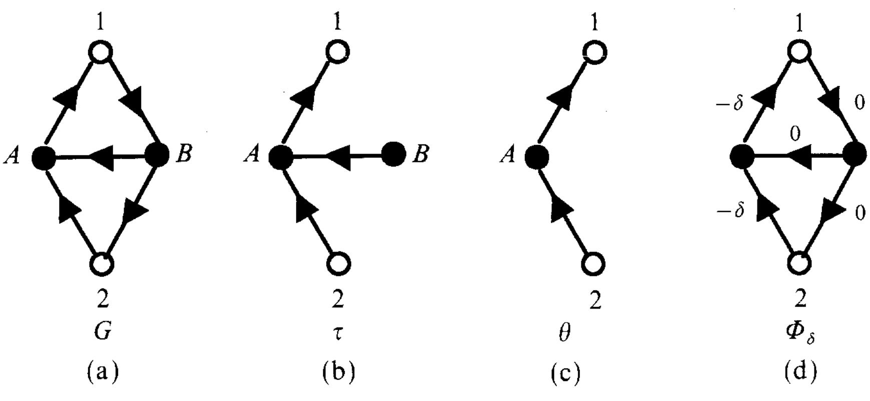

Fig. 1. (a) A graph G whose edges are given arbitrary directing, indicated by the arrows. The roots 1 and 2 represented by open circles and unrooted vertices by solid circles. (b) An arbitrary spanning tree τ of G and (c) its corresponding path θ between the roots. By adding the flow Φ_δ (d) to unrooted flows, one generates rooted δ-flows.

can generate from the unrooted mod-6 flows shown in the first column of Fig. 2 the corresponding rooted mod-6 1-flows and rooted mod-6 3-flows drawn in the second and third columns, respectively. These were obtained from the unroted mod-6 flows by adding the flow Φ_δ shown in Fig. 1d with $\delta = 1$ and $\delta = 3$, respectively.

We begin the proof of (3.4) by noting that similarly to the above procedure, one can generate the rooted mod-$2n$ $(\alpha + 2)$-flows contributing to $N_\alpha(1, 2; G)$ by adding, to each of the rooted mod-$2n$ α-flows which are generated by $N_\alpha(1, 2; G)$, a flow Φ_2 equal to 2 on the unique path θ from 1 to 2 and zero on all other edges. This provides a bijective mapping between the flows of $N_\alpha(1, 2; G)$ and those of $N_{\alpha+2}(1, 2; G)$. Notice that for λ even (which is the case we are considering here with $\lambda = 2n$), this procedure cannot change the value of the flow on any edge from odd to even.

Now let us consider an n-component cubic cluster in which, for notational simplicity, we shall assume that $t_e = t$ for all edges e. By the above construction the powers of $t(1)$ [which are, in the cubic model, associated with the odd flows according to Eq. (3.5a)] must be the same for any rooted mod-$2n$ α-flow and its corresponding rooted mod-2n $(\alpha + 2)$-flow. Also, when $t(1) = t(2) = t$ we must regain the Potts model formulas. It follows that, since any term $[t(1)]^k [t(2)]^l$ $(k, l = 0, 1,..., \varepsilon)$ which appears in $N_\alpha(1, 2; G)$ for the cubic model becomes t^{k+l} in $N(1, 2; G)$ for the Potts model, the power of $t(2)$ for corresponding flows is different if and only if the power of t is different for the Potts model. Considering that, for $\alpha > 0$, (i) $N(1, 2; G)$ is independent of the external flow for the Potts model, and (ii) the addition of the flow Φ_2 does not change the number of edges with odd values of flow, we conclude that the changes in the power of t for different mod-$2n$ flows with a fixed number k of edges on which their values are odd compensate in such a way as to maintain the same sum. This induces a compensation in the powers of $t(2)$ for the cubic model in such a way that the term

$$[t(1)]^k \sum_{l=0}^{\varepsilon - k} a_l [t(2)]^l$$

is the same for both $N_\alpha(1, 2; G)$ and $N_{\alpha+2}(1, 2; G)$. In the last two examples of Fig. 2 we show the compensation between the terms $[t(1)]^2$ and $[t(1)]^2 [t(2)]^2$ which occur in $N_1(1, 2; G)$ and $N_3(1, 2; G)$ for the 3-component cubic model on the graph G of Fig. 1a. In general, as this compensation happens for any power k $(k = 0, 1,..., \varepsilon)$ of $t(1)$, then it follows that $N_\alpha(1, 2; G) = N_{\alpha+2}(1, 2; G)$, leading thus to Eqs. (3.4).

The combination of Eqs. (3.4) and (2.9a) shows that the equivalent vector transmissivity has only two distinct components.

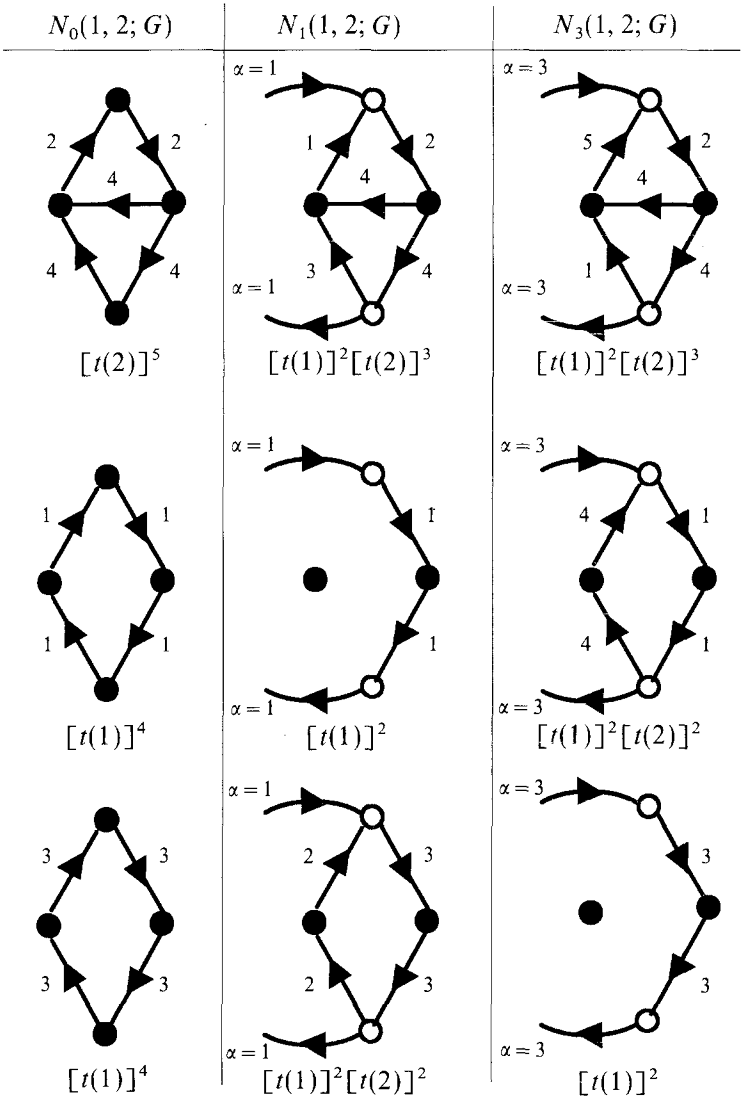

Fig. 2. Examples of unrooted mod-6 flows (first column) and its corresponding rooted 1-flows (second column) and rooted 3-flows (third column) on the graph G of Fig. 1a. These rooted flows were obtained from the unrooted ones by adding the flow Φ_δ of Fig. 1d for $\delta = 1$ and 3, respectively. A missing edge indicates that the value of the flow on it is zero. α represents the external flow in at the root 1 and out at the root 2. To each edge with a nonzero even (odd) value of flow is associated a transmissivity $t(2)$ $[t(1)]$. Below each α-flow the corresponding term contributing to the generating function $N_\alpha(1, 2; G)$ is given.

4. GRAPH REDUCTION EQUATIONS OF THE SBCM

The main purpose of the SBCM is to calculate the flow vector for a graph G (and hence the partition function and pair correlation functions) in terms of those for "smaller" graphs. Three methods of reducing the size of a graph are used in the SBCM: (i) splitting into pieces at articulation vertices; (ii) replacement of subgraphs attached at only two vertices by effective edges; (iii) removal of (effective) edges through the use of the effective break-collapse equation.

The graph reduction equations for the n-component cubic model associated with the above procedures will now be derived from those for the $Z(2n)$ model.

4.1. Splitting of Articulated Graphs

Suppose that a two-rooted graph G is composed of two subgraphs G_1 and G_2 which intersect only at the articulation point i (see Fig. 3). Two cases can arise: (a) both roots 1 and 2 belong to one of the subgraphs, say G_1 (Fig. 3a); (b) the root 1 belongs to, say, G_1 and 2 is in G_2 (Fig. 3b). In case (b) if $i \neq 1$ or 2, then G_1 and G_2 are said to be in series.

(a) Both roots in G_1. Equation (3.2) of ZF gives

$$N_\alpha(1, 2; G) = N_\alpha(1, 2; G_1)\, D(G_2) \qquad (\alpha = 0, 1, 2) \tag{4.1}$$

(b) G_1 and G_2 are in series. It follows from (3.3) of ZF that

$$N_\alpha(1, 2; G) = N_\alpha(1, i; G_1)\, N_\alpha(i, 2; G_2) \qquad (\alpha = 0, 1, 2) \tag{4.2}$$

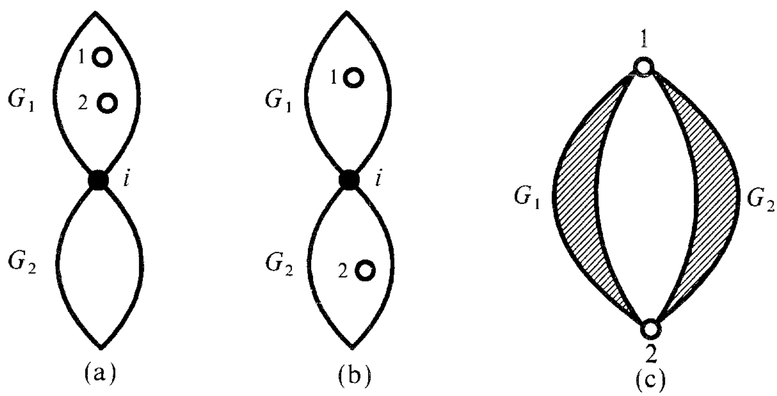

Fig. 3. Pictorial representations of two graphs G_1 and G_2 which (a, b) share an articulation vertex i or (c) are in parallel. In panel (b) the graphs G_1 and G_2 are in series.

which, for two ordinary edges ($G_1 = e_1$ and $G_2 = e_2$) recovers the result of Tsallis *et al.*[10]

4.2. Parallel Combination of Graphs

Now let us consider a two-rooted graph G with is the union of two subgraphs G_1 and G_2 which intersect only at roots 1 and 2 (see Fig. 3c). In this case, G_1 and G_2 are said to be in parallel. Using Eqs. (3.4) of ZF and Eqs. (3.4) here, we find

$$\begin{aligned} D(G) = {} & D(G_1)\, D(G_2) + nN_1(1, 2; G_1)\, N_1(1, 2; G_2) \\ & + (n-1)\, N_2(1, 2; G_1)\, N_2(1, 2; G_2) \end{aligned} \tag{4.3a}$$

$$\begin{aligned} N_1(1, 2; G) = {} & D(G_1)\, N_1(1, 2; G_2) + D(G_2)\, N_1(1, 2; G_1) \\ & + (n-1)[N_2(1, 2; G_1)\, N_1(1, 2; G_2) \\ & + N_1(1, 2; G_1)\, N_2(1, 2; G_2)] \end{aligned} \tag{4.3b}$$

and

$$\begin{aligned} N_2(1, 2; G) = {} & D(G_1)\, N_2(1, 2; G_2) + D(G_2)\, N_2(1, 2; G_1) \\ & + nN_1(1, 2; G_1)\, N_1(1, 2; G_2) \\ & + (n-2)\, N_2(1, 2; G_1)\, N_2(1, 2; G_2) \end{aligned} \tag{4.3c}$$

Equations (4.3) particularized for two ordinary edges e_1 and e_2 in parallel give

$$t_p(1) \equiv \frac{N_1(1, 2; G)}{D(G)} = \frac{t_1(1) + t_2(1) + (n-1)[t_1(2)\, t_2(1) + t_1(1)\, t_2(2)]}{1 + nt_1(1)\, t_2(1) + (n-1)\, t_1(2)\, t_2(2)} \tag{4.4a}$$

and

$$t_p(2) \equiv \frac{N_2(1, 2; G)}{D(G)} = \frac{t_1(2) + t_2(2) + nt_1(1)\, t_2(1) + (n-2)\, t_1(2)\, t_2(2)}{1 + nt_1(1)\, t_2(1) + (n-1)\, t_1(2)\, t_2(2)} \tag{4.4b}$$

which agrees with the parallel algorithm of Tsallis *et al.*[10]

Equations (4.3) can be written in a factorized form similar to the series equation as (ZF)

$$\tilde{N}_\beta(1, 2; G) = \tilde{N}_\beta(1, 2; G_1)\, \tilde{N}_\beta(1, 2; G_2) \tag{4.5}$$

where the discrete Fourier transforms $\tilde{N}_\beta$ are

$$\tilde{D}(G) = D(G) + nN_1(1, 2; G) + (n-1)\, N_2(1, 2; G) \tag{4.6a}$$

$$\tilde{N}_n(1, 2; G) = D(G) - nN_1(1, 2; G) + (n-1)\, N_2(1, 2; G) \tag{4.6b}$$

and

$$\tilde{N}_\alpha(1, 2; G) = D(G) - N_2(1, 2; G) \qquad (\forall \alpha \neq 0 \text{ or } n) \tag{4.6c}$$

When G is a single ordinary edge e connecting 1 and 2, then $\tilde{N}_\beta/\tilde{N}_0$ is equal to the dual variable $[t(\beta)]^{\mathrm{D}}$ of $t(\beta)$ defined for the $Z(\lambda)$ model in ref. 18. The dual vector transmissivity for the n-component cubic model is therefore given by

$$[t_e(n)]^{\mathrm{D}} \equiv \frac{\tilde{N}_n}{\tilde{D}} = \frac{1 - nt_e(1) + (n-1)\, t_e(2)}{1 + nt_e(1) + (n-1)\, t_e(2)} = e^{-2nK_e} \tag{4.7a}$$

and for $\alpha \neq 0$ or n

$$[t_e(\alpha)]^{\mathrm{D}} \equiv \frac{\tilde{N}_\alpha}{\tilde{D}} = \frac{1 - t_e(2)}{1 + nt_e(1) + (n-1)\, t_e(2)} = e^{-n(K_e + L_e)} \tag{4.7b}$$

which are, respectively, the variables x_β and x_α used in the model of Domany and Riedel[6] specialized to the cubic Hamiltonian. Combining Eqs. (4.5) and (4.7), we get the following alternative equation for two ordinary edges in parallel:

$$[t_p(\beta)]^{\mathrm{D}} = [t_1(\beta)]^{\mathrm{D}}\, [t_2(\beta)]^{\mathrm{D}} \qquad (\forall \beta) \tag{4.8}$$

4.3. Replacement of a Subgraph by an Effective Edge

Let us consider a two-rooted graph G which is the union of two subgraphs H and L which intersect at only two vertices, i and j. Furthermore, both roots 1 and 2 belong to H (see Fig. 4) with the possibility that i and/or j are rooted. When both i and j are rooted, then L and H are in parallel and we recover the results of Section 4.2.

In ZF it was proved, through the use of flows, that one can replace the subgraph L by a single effective edge e_L having an effective flow vector equal to the flow vector of L. This result can be stated for the cubic model as

$$N_\alpha(1, 2; H \cup L) = N_\alpha(1, 2; H \cup e_L) \tag{4.9a}$$

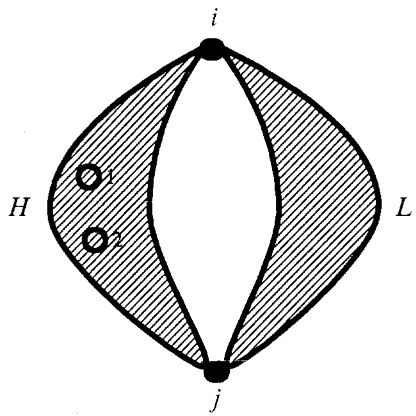

Fig. 4. Pictorial representation of a two-reducible graph $G = H \cup L$ with the roots 1 and 2 in H. Each subgraph is represented by a half-moon shape.

with

$$N_\alpha(i, j; e_L) = N_\alpha(i, j; L) \qquad (\alpha = 0, 1, 2) \tag{4.9b}$$

$N_\alpha(i, j; e_L)$ can be calculated through the SBCM or by performing the partial trace over the internal vertices of L as mentioned in Section 2.2. Equation (4.9a) may be repeatedly applied as long as there are further subgraphs which satisfy the above conditions on L. Also, the subgraphs replaced may themselves contain effective edges.

The replacement of a subgraph by an effective edge is an essential step in the SBCM. The subgraphs to be replaced are considered to be of three types: (i) two (effective) edges in series, (ii) two (effective) edges in parallel, (iii) subgraphs which are not combinations of series and/or parallel (effective) edges. The latter replacement will be called, as in ZF, a nonreducible subgraph replacement. The search for suitable subgraphs is performed in this order.

4.4. The Effective Break-Collapse Equation

When no more subgraph replacements can be made, then one must apply the effective break-collapse equation. Combining Eq. (3.16) of ZF with Eqs. (3.4) here, we get the following effective break-collapse equation for the cubic model:

$$\begin{aligned} N_\alpha(1, 2; G) = {} & [D_{\text{eff}} - N_{2\text{eff}}]\, N_\alpha(1, 2; G_f^\delta) + N_{1\text{eff}} N_\alpha(1, 2; G_f^\gamma) \\ & + (N_{2\text{eff}} - N_{1\text{eff}})\, N_\alpha^{ev}(1, 2; f; G) \qquad (\alpha = 0, 1, 2) \end{aligned} \tag{4.10a}$$

where

$$N_\alpha^{ev}(1, 2; f; G) = N_{\alpha 0}(1, 2; f; G) + N_{\alpha 2}(1, 2; f; G) + N_{\alpha 4}(1, 2; f; G) + \cdots + N_{\alpha, 2n-2}(1, 2; f; G) \tag{4.10b}$$

In (4.10a), G_f^δ and G_f^γ are, respectively, the deleted and contracted graphs obtained from G by deleting a chosen (effective) edge f and contracting it (i.e., identifying the endpoints of f in G_f^δ). Here D_{eff}, $N_{1\text{eff}}$, and $N_{2\text{eff}}$ are the components of the flow vector of the (effective) edge f, and $N_{\alpha\beta}(1, 2; f; G)$ is the generating function for rooted mod-$2n$ α -flows having a fixed flow β in the edge f. Such an edge will be called, as in ZF, a *frozen edge*.

The components of the flow vectors for the deleted and contracted graphs are related to $N_{\alpha\beta}$ through (ZF)

$$N_\alpha(1, 2; G_f^\delta) = N_{\alpha 0}(1, 2; f; G) \tag{4.11}$$

and

$$N_\alpha(1, 2; G_f^\gamma) = \sum_{\beta=0}^{2n-1} N_{\alpha\beta}(1, 2; f; G) \tag{4.12}$$

In order words, to delete an edge f is equivalent to having a frozen edge f on which the flow has value zero, and to contract f is equivalent to summing over all possible values of the flow for this edge.

Now consider the relation between N_α^{ev} and the flow vector for G. Using the relationship between N_α and $N_{\alpha\beta}$ [see Eq. (3.10) of ZF] with $t_f(\beta) = N_{\beta\text{eff}}$ particularized for the cubic model, namely

$$\begin{aligned} N_\alpha(1, 2; G) = D_{\text{eff}} N_{\alpha 0}(1, 2; f; G) &\\ + N_{1\text{eff}}[N_{\alpha 1}(1, 2; f; G) + N_{\alpha 3}(1, 2; f; G) &\\ + \cdots + N_{\alpha, 2n-1}(1, 2; f; G)] &\\ + N_{2\text{eff}}[N_{\alpha 2}(1, 2; f; G) + N_{\alpha 4}(1, 2; f; G) &\\ + \cdots + N_{\alpha, 2n-2}(1, 2; f; G)] & \end{aligned} \tag{4.13}$$

and comparing it with Eq. (4.10b), it follows that

$$N_\alpha^{ev}(1, 2; f; G) = N_\alpha(1, 2; G)|_{\substack{N_{1\text{eff}} = 0 \\ D_{\text{eff}} = N_{2\text{eff}} = 1}} \tag{4.14}$$

The right-hand side of Eq. (4.14) is similar to the flow vector $N_\alpha^{bc}(1, 2; G)$ for the $Z(4)$ model defined for the graph G with a chosen edge

f "precollapsed."[13] However, for the $Z(4)$ model, $N_{\alpha}^{bc}(1, 2; G)$ is the generating function for rooted mod-4 α-flows having value 0 or 2 on the edge f (see ZF), while here $N_{\alpha}^{ev}(1, 2; f; G)$ is the generating function for rooted mod-$2n$ α-flows having value $0, 2, 4,..., 2n-2$ of f. In this condition f will be called an *even frozen edge.*

If f is an ordinary edge, then Eq. (4.10a) recovers a conjectured result (C. Tsallis, private communication).

In the SBCM, Eq. (4.10a) is applied recursively so that the flow vector of G may be equal to that with several even frozen edges. In this case, N_{α}^{ev} satisfies an effective break-collapse equation similar to Eq. (4.10a). The latter equation is applied as many times as needed to arrive at either graphs with just two vertices, or graphs with *all* edges even frozen. For such graphs (which we will denote by G_{ev}), $N_{\alpha}(1, 2; G_{ev})$ is the number of rooted mod-$2n$ α-flows with the constraint that the flow on all edges must be even frozen. Such flows will be called, as in ZF, *even rooted mod-2n α-flows.* Following along the same lines as in the proof of Eqs. (4.2) of ZF, one can easily show that

$$N_1(1; 2; G_{ev}) = 0 \tag{4.15a}$$

$$N_2(1, 2; G_{ev}) = n^{c(G_{ev})} \gamma_{12}(G_{ev}) \tag{4.15b}$$

$$D(G_{ev}) = n^{c(G_{ev})} \tag{4.15c}$$

where $c(G_{ev})$ is the number of independent cycles in G_{ev}; $\gamma_{12}(G_{ev})$ is 1 if the roots are connected and zero otherwise.

It is worthwhile stressing that, unlike the effective break-collapse equation for the $Z(\lambda)$ model, Eq. (4.10a) allows the calculation of the flow vector as a function of n rather than for a specified value of n. In the case of the $Z(\lambda)$ model, the application of the break-collapse equation generates, besides the broken and collapsed graphs, a further $\lambda - 3$ graphs, while in the cubic model it generates *only one further graph independent of the value of n.*

4.5. Particular Cases

Now let us show that our graph reduction equations reproduce correctly the expected results in different particular cases of the cubic model.

4.5.1. $n = 1$ (Ising Model). For $n = 1$, the vector transmissivity has only one component given by [Eq. (3.3a)]

$$t_e(1) = \tanh K_e \tag{4.16}$$

which is the thermal transmissivity t_e defined for the Ising model with coupling constant K_e. Our respective graph reduction equations (4.1), (4.2), (4.3a), and (4.3b) reduce, for $n=1$, to Eqs. (4.17a), (4.17b), (4.14b), and (4.14a) of PF3 particularized for a two-rooted Ising cluster. From Eqs. (4.10b) and (4.11) it follows that, for $\alpha=0, 1$,

$$N_\alpha^{ev}(1, 2; f; G)|_{n=1} = N_\alpha(1, 2; G_f^\delta)|_{n=1} \tag{4.17}$$

which combined with Eq. (4.10a) leads to the effective break-collapse equation [see Eqs. (4.13) of PF3] for the Ising model.

4.5.2. $n=2$ (Symmetric Ashkin–Teller Model). Aharony[2] showed that, for $n=2$, the Hamiltonian of the cubic model [Eq. (2.3)] can be written in terms of two coupled Ising variables in the same form as that of the symmetric Ashkin and Teller model.[21] The Hamiltonian of the latter model is identical to that for the $Z(4)$ model described in Eq. (1) of Mariz *et al.*[13] with coupling constants $K_1=K$ and $K_2=L/2$.

The components $t_e(1)$ and $t_e(2)$ of the vector transmissivity [Eq. (3.3)] become, for $n=2$, identical to the respective transmissivities t_1 and t_2 defined in Eqs. (2a) and (2b) of Mariz *et al.*,[13] where $K_1=K$ and $K_2=L/2$. One can easily see that our SBCM graph reduction equations reduce, for $n=2$, to those derived in ZF for the $Z(4)$ model, as it should be.

4.5.3. $K_e=L_e$ (2n-State Potts Model). The case $K_e=L_e$ corresponds to a $2n$-state Potts model with coupling constant $2nK_e$ (see ref. 2). In this case, Eqs. (3.3) become

$$t_e(1) = t_e(2) = \frac{1-e^{-2nK_e}}{1+(2n-1)e^{-2nK_e}} \tag{4.18}$$

which is the thermal transmissivity [see Eq. (1) of ref. 22] of a $2n$-state Potts model. Using the fact that, for the Potts model [see Eq. (2.12a)],

$$N_1(1, 2; G) = N_2(1, 2; G) = N(1, 2; G) \tag{4.19}$$

one can easily show that our graph reduction equations reproduce the expected results (see PF3).

4.5.4. $K_e=0$ (n-State Potts Model). Setting $K_e=0$ in Eq. (2.3) leads to the Hamiltonian of an n-state Potts model with coupling constant nL_e. In this case Eqs. (3.3) become

$$t_e(1) = 0 \tag{4.20a}$$

$$t_e(2) = \frac{1-e^{-nL_e}}{1+(n-1)e^{-nL_e}} = t_e \tag{4.20b}$$

where t_e is the thermal transmissivity of an n-state Potts model. From Eqs. (4.20a) and (3.5a) we conclude that $N_\alpha(1, 2; G)$ becomes, in the considered case, the generating function for *even* rooted mod-$2n$ α-flows. From conservation of mod-$2n$ flows it follows then, similarly to Eq. (4.15a), that

$$N_1(1, 2; G) = 0 \tag{4.21}$$

Furthermore, for $\alpha = 2\beta$, there is a bijective correspondence between the even rooted mod-$2n$ α-flows and the unrestricted rooted mod-n β-flows obtained by replacing edges with flow $2l$ by edges with flow l ($l = 0, 1, ..., n-1$). Consequently, in this case

$$N_2(1, 2; G) = N(1, 2; G) \tag{4.22}$$

where $N(1, 2; G)$ is the generating function for the rooted mod-n flows in the n-state Potts model.

Combining relations (4.21), (4.22), and (4.12), one can easily prove that, for $K_e = 0$, all our graph reduction equations for $D(G)$ and $N_2(1, 2; G)$ reduce to those obtained for the Potts model (PF3).

5. SBCM FOR THE n-COMPONENT CUBIC MODEL

In this section we describe the modifications of the SBCM algorithm for the Potts model (PF3) necessary for treating the cubic model. We also illustrate the SBCM for the n-component cubic model using the Wheatstone bridge cluster.

5.1. The SBCM Algorithm

The SBCM algorithm for the Potts model described in PF3 contains a recursive procedure T which executes the operations of splitting into pieces, replacement of (effective) edges in series or in parallel by a single efective edge, and the operation of nonreducible subgraph replacement as long as possible. It then applies the effective break-collapse equation. The use of this equation as well as the nonreducible subgraph replacement require calls to T; thus, the algorithm is recursive. The terminal condition for the procedure arises when a graph with only two vertices is arrived at, in which case the equivalent transmissivity is calculated by the parallel reduction equation. The SBCM algorithm for the cubic model differs from that for the Potts model in the following respects:

(i) Instead of associating to each edge $e = [i, j]$ the numerator N_e and denominator D_e of the effective thermal transmissivity of e, we associate the components $N_0(i, j; e)$, $N_1(i, j; e)$, and $N_2(i, j; e)$ of the effective flow vector of the edge e.

(ii) The effective break-collapse equation must be replaced by Eq. (4.10a), which demands the calculation of $N_\alpha^{ev}(1, 2; f; G)$. This may be accomplished by replacing step (IId4) of the algorithm by a loop with a further call to T for the graph G with an even frozen edge f. The series and parallel reduction equations work without modification provided we set $t_f(0) = t_f(2) = 1$ and $t_f(1) = 0$.

(iii) In the selection of the (effective) edge to be deleted and contracted [step (IId1) of the algorithm], this must now not be an even frozen edge.

(iv) A further terminal step must be added before the terminal condition mentioned in (IIe) of PF3. This refers to a graph with more than two vertices, the edges of which are all even frozen. In this case, there is no need for further applications of the effective break-collapse equation, since the flow vector of the current graph is given by Eqs. (4.15).

5.2. An Illustration of the SBCM

Now let us illustrate the SBCM by calculating the equivalent vector transmissivity of the Wheatstone bridge graph G of Fig. 5. We consider only the case when all edges have the same vector transmissivity t.

Since G has five edges, it is necessary to apply the effective break-collapse equation [Eq. (4.10a)] five times, arriving thus at the graph G_{ev} of Fig. 5, whose edges are all even frozen. Figure 5 shows the "tree" of graphs generated in the SBCM where the edges to be deleted and contracted were chosen in the following sequence: e_5, e_2, e_1, e_3, and e_4. For the sake of simplicity, the further graphs resulting from the replacement of edges (which can be even frozen or not) in series and/or parallel by effective edges are not included in Fig. 5. The branching into two graphs refers to the splitting of articulated graphs, while the one into three graphs results from the application of the effective break-collapse equation. The effective flow vectors for the terminal graphs shown in Fig. 5 are the following:

$$N_\alpha(1, 2; G_{11}) = n \qquad (\alpha = 0, 2) \tag{5.1a}$$

$$N_1(1, 2; G_{11}) = 0 \tag{5.1b}$$

$$N_\alpha(1, 2; G_{12}) = N_\alpha(1, 2; G_{ev}) = n^2 \qquad (\alpha = 0, 2) \tag{5.2a}$$

$$N_1(1, 2; G_{12}) = N_1(1, 2; G_{ev}) = 0 \tag{5.2b}$$

$$N_\alpha(1, 2; G_{10}) = 1 + (n - 1)\, t(2) \qquad (\alpha = 0, 2) \tag{5.3a}$$

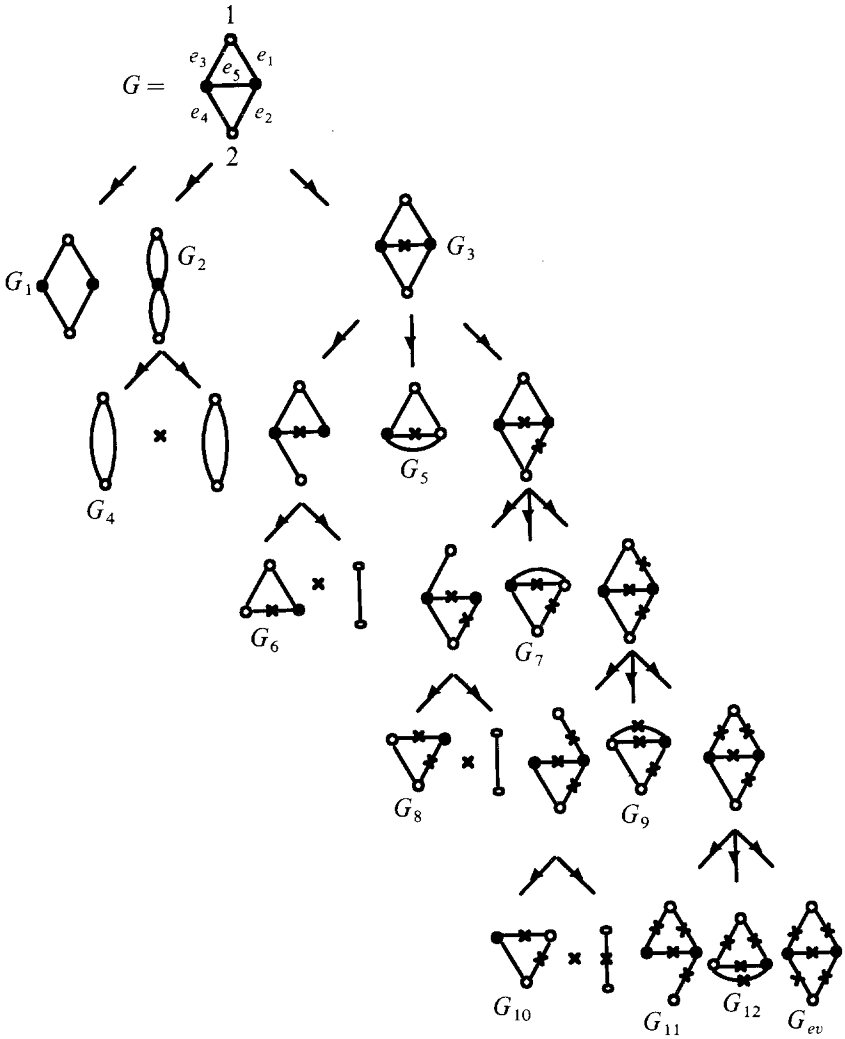

Fig. 5. A schematic representation of the SBCM calculation of $N(1, 2; G)$ for the Wheatstone bridge graph. The further steps are not shown for graphs which are combinations of series and/or parallel edges. The splitting of an articulated graph is indicated by the sign $\times$ between the two subgraphs. The crossed line represents an even frozen edge whose vector transmissivity is given by $t(0) = t(2) = 1$ and $t(1) = 0$. The vector transmissivity associated with any other edge is t.

$$N_1(1, 2; G_{10}) = 0 \tag{5.3b}$$

$$N_\alpha(1, 2; G_9) = n + n(n-1)\, t(2) \qquad (\alpha = 0, 2) \tag{5.4a}$$

$$N_1(1, 2; G_9) = n^2 t(1) \tag{5.4b}$$

$$N_\alpha(1, 2; G_8) = 1 + (n-1)\, t(2) \qquad (\alpha = 0, 2) \tag{5.5a}$$

$$N_1(1, 2; G_8) = nt(1) \tag{5.5b}$$

$$N_\alpha(1, 2; G_7) = 1 + 2(n-1)\, t(2) + (n-1)^2[t(2)]^2 \qquad (\alpha = 0, 2) \tag{5.6a}$$

$$N_1(1, 2; G_7) = n^2[t(1)]^2 \tag{5.6b}$$

$$N_0(1, 2; G_6) = 1 + (n-1)[t(2)]^2 \tag{5.7a}$$

$$N_1(1, 2; G_6) = t(1) + (n-1)\, t(1)\, t(2) \tag{5.7b}$$

$$N_2(1, 2; G_6) = 2t(2) + (n-2)[t(2)]^2 \tag{5.7c}$$

$$\begin{aligned} N_0(1, 2; G_5) = 1 &+ (n-1)\, t(2) + (n-1)[t(2)]^2 \\ &+ (n-1)^2[t(2)]^3 + n^2[t(1)]^3 \end{aligned} \tag{5.8a}$$

$$\begin{aligned} N_1(1, 2; G_5) = t(1) &+ 2(n-1)\, t(1)\, t(2) + n[t(1)]^2 \\ &+ n(n-1)[t(1)]^2\, t(2) + (n-1)^2\, t(1)[t(2)]^2 \end{aligned} \tag{5.8b}$$

$$N_2(1, 2; G_5) = 2t(2) + (3n-4)[t(2)]^2 + (n-1)(n-2)[t(2)]^3 + n^2[t(1)]^3 \tag{5.8c}$$

$$N_0(1, 2; G_4) = 1 + n[t(1)]^2 + (n-1)[t(2)]^2 \tag{5.9a}$$

$$N_1(1, 2; G_4) = 2t(1) + 2(n-1)\, t(1)\, t(2) \tag{5.9b}$$

$$N_2(1, 2; G_4) = 2t(2) + n[t(1)]^2 + (n-2)[t(2)]^2 \tag{5.9c}$$

Combining the above expresions together with the appropriate graph reduction equation of Section 4, we get that

$$N_0(1, 2; G_1) = 1 + n[t(1)]^4 + (n-1)[t(2)]^4 \tag{5.10a}$$

$$N_1(1, 2; G_1) = 2\{[t(1)]^2 + (n-1)[t(1)]^2[t(2)]^2\} \tag{5.10b}$$

$$N_2(1, 2; G_1) = 2[t(2)]^2 + n[t(1)]^4 + (n-2)[t(2)]^4 \tag{5.10c}$$

$$N_0(1, 2; G_2) = \{1 + n[t(1)]^2 + (n-1)[t(2)]^2\}^2 \tag{5.11a}$$

$$N_1(1, 2; G_2) = 4[t(1)]^2\{1 + (n-1)\, t(2)\}^2 \tag{5.11b}$$

$$N_2(1, 2; G_2) = \{2t(2) + n[t(1)]^2 + (n-2)[t(2)]^2\}^2 \tag{5.11c}$$

$$N_0^{ev}(1, 2; e_5; G_3) = 1 + 2(n-1)[t(2)]^2 + n^2[t(1)]^4 + (n-1)^2[t(2)]^4 \tag{5.12a}$$

$$N_1^{ev}(1, 2; e_5; G_3) = 2[t(1)]^2 + 4(n-1)[t(1)]^2\, t(2) + 2(n-1)^2[t(1)]^2[t(2)]^2 \tag{5.12b}$$

$$N_2^{ev}(1, 2; e_5; G_3) = 4[t(2)]^2 + 4(n-2)[t(2)]^3 + n^2[t(1)]^4 + (n-2)^2[t(2)]^4 \tag{5.12c}$$

Combining Eqs. (5.10)–(5.12) with the effective break-collapse equation for $f = e_5$, namely [see Eq. (4.10a)]

$$N_\alpha(1, 2; G) = [1 - t(2)]\, N_\alpha(1, 2; G_1) + t(1)\, N_\alpha(1, 2; G_2) + [t(2) - t(1)]\, N_\alpha^{ev}(1, 2; e_5; G_3) \tag{5.13}$$

we finally arrive at the flow vector of G:

$$\begin{aligned} N_0(1, 2; G) = {} & 1 + 2n[t(1)]^3 + 2(n-1)[t(2)]^3 \\ & + n[t(1)]^4 + (n-1)[t(2)]^4 \\ & + (n-1)(n-2)[t(2)]^5 + 2n(n-1)[t(1)]^3[t(2)]^2 \\ & + n(n-1)[t(1)]^4\, t(2) \end{aligned} \tag{5.14a}$$

$$\begin{aligned} N_1(1, 2; G) = {} & 2[t(1)]^2 + 2[t(1)]^3 + 6(n-1)[t(1)]^2[t(2)]^2 \\ & + 2(n-1)(n-2)[t(1)]^2[t(2)]^3 \\ & + 4(n-1)[t(1)]^3\, t(2) + 2(n-1)^2[t(1)]^3[t(2)]^2 \end{aligned} \tag{5.14b}$$

$$\begin{aligned} N_2(1, 2; G) = {} & 2[t(2)]^2 + 2[t(2)]^3 + n[t(1)]^4 \\ & + 5(n-2)[t(2)]^4 + 4n[t(1)]^3\, t(2) \\ & + 2n(n-2)[t(1)]^3[t(2)]^2 \\ & + n(n-1)[t(1)]^4[t(2)] + (n-2)(n-3)[t(2)]^5 \end{aligned} \tag{5.14c}$$

Combining Eqs. (5.14) with the definitions of $t(1)$ and $t(2)$ [Eqs. (3.3)], one obtains an equivalent vector transmissivity which has effective coupling constants K_{eff} and L_{eff} equal to the respective renormalized coupling constants K' and NL' of Tsallis *et al.*[10]

Notice that Eqs. (5.14) recover, for all the particular cases considered in Section 4.5, the expected results (see PF3 and ref. 13).

6. CONCLUSIONS

We have generalized to the n-component cubic model the subgraph break-collapse method (SBCM) of the Potts model which we presented elsewhere. While in the latter model the equivalent transmissivity was a scalar, it becomes a two-dimensional vector for all values of n in the cubic model. The effective break-collapse equation involves also, besides the broken and collapsed graphs which appear in the Potts model, a graph with an edge on which the value of the flow is even. We have called the latter an even frozen edge.

Our graph reduction equations were derived from those we developed recently for the $Z(\lambda)$ model. However, the SBCM algorithm for the cubic model differs from that for the $Z(2n)$ in the following aspects: (i) it contains graphs with even frozen edges instead of frozen edges having fixed flows; (ii) its effective break-collapse equation generates only three flow vectors for all values of n instead of $(2n-1)$; (iii) it gives the equivalent vector transmissivity as a function of n rather than for a fixed value of n; (iv) it requires more iterations, since the terminal condition refers to graphs with all edges even frozen rather than a number of frozen edges equal to the number of independent cycles.

An even frozen edge is equal, for $n=2$, to the precollapsed edge which appears in the break-collapse method (BCM) for the $Z(4)$ model.[13] In this case, our algorithm becomes similar to the BCM, but with the important difference that we include nonreducible subgraph replacements.

ACKNOWLEDGMENTS

We wish to thank C. Tsallis and A. M. Mariz for helpful discussions and for communicating the break-collapse formulas for the cubic model prior to publication. One of us (A.C.N.dM.) is grateful for the hospitality and financial support of RHBNC, and also acknowledges the research fellowship of CNPq and the financial suport of SERC.

REFERENCES

1. D. Kim, P. M. Levy, and L. F. Uffer, *Phys. Rev. B* **12**:989 (1975).
2. A. Aharony, *J. Phys. A: Math. Gen.* **10**:389 (1977).
3. D. Kim and P. M. Levy, *Phys. Rev. B* **12**:5105 (1975).
4. D. Kim, P. M. Levy, and J. J. Sudano, *Phys. Rev. B* **13**:2054 (1976).
5. H. J. Hilhorst, *Phys. Lett.* **56A**:153 (1976).
6. E. Domany and E. K. Riedel, *Phys. Rev. B* **19**:5817 (1979).
7. B. Nienhuis, E. K. Riedel, and M. Schick, *Phys. Rev. B* **27**:5625 (1983).
8. R. Badke, P. Reinicke, and V. Rittenberg, *J. Phys. A: Math. Gen.* **18**:653 (1985).
9. R. Badke, *Phys. Lett. A* **119**:365 (1987).

10. C. Tsallis, A. M. Mariz, A. Stella, and L. R. da Silva, *J. Phys. A: Math. Gen.* **22** (1990).
11. A. C. N. de Magalhães and J. W. Essam, *J. Phys. A: Math. Gen.* **22**:2549 (1989).
12. A. C. N. de Magalhães and J. W. Essam, *J. Phys. A: Math. Gen.* **21**:473 (1988).
13. A. M. Mariz, C. Tsallis, and P. Fulco, *Phys. Rev. B* **32**:6055 (1985).
14. A. M. Mariz, A. C. N. de Magalhães, L. R. da Silva, and C. Tsallis, to be published.
15. C. Tsallis, *Phys. Rep. Physica A* **319** (1990).
16. E. P. da Silva, C. Tsallis, and A. M. Mariz, to be published.
17. N. Biggs, *Math. Proc. Camb. Phil. Soc.* **80**:429 (1976).
18. N. Biggs, *Interaction Models* (Cambridge University Press, Cambridge, 1977).
19. F. C. Alcaraz and C. Tsallis, *J. Phys. A: Math. Gen.* **15**:587 (1982).
20. J. W. Essam and C. Tsallis, *J. Phys. A: Math. Gen.* **19**:409 (1986).
21. J. Ashkin and E. Teller, *Phys. Rev.* **64**:178 (1943).
22. C. Tsallis and S. V. F. Levy, *Phys. Rev. Lett.* **47**:950 (1981).